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The genus of the GRAY graph is 7

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Dedicated to Arthur T. White for his work on the threefold connections between graphs, groups and surfaces

Abstract

Using the genus embedding of the Cartesian product of three triangles we prove one can embed the smallest cubic semisymmetric graph on 54 vertices, the so-called Gray graph, in the orientable surface of genus 7, and we prove that such an embedding is optimal.

1. Introduction

Using an old result on the genus of the Cartesian product of three triangles, compare [5, 6, 15], we prove that the genus of the Gray graph $\mathcal{G}$ is indeed 7 by exploiting its relation to the face-bicolored minimal genus embedding of the Cartesian product of three triangles M. The two graphs are related via a (27_3) configuration, $\mathcal{G}$ being its Levi graph and M being its Menger graph. By using a covering graph technique we can show that the 6-valent Menger graph of the dual configuration contains the 4-valent Holt graph H, the minimal $1/2$-arc transitive graph, as a spanning subgraph.

2. Incidence structures and configurations

An *incidence structure* $\mathcal{C}$ is a triple $\mathcal{C} = (P, B, I)$ where P is the set of points, B is the set of lines, and $I \subseteq P \times B$ is the incidence relation. The elements of I are called *flags*. The bipartite incidence graph $L(\mathcal{C})$ with black vertices P, white vertices B and edges I

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is known as the *Levi graph* of $\mathcal{C}$. We form the *Menger graph* $M(\mathcal{C})$ of the incidence structure $\mathcal{C}$; its vertices are the points and two vertices are joined by an edge when the points belong to the same line of $\mathcal{C}$ [8]. A (v_r, b_k) *configuration* is an incidence structure $\mathcal{C} = (P, B, I)$ in which two lines meet in at most one point such that $v = |P|, b = |B|$, there are r lines through a point and there are k points on a line. It follows easily that $vr = |I| = bk$. Note that the Levi graph of a (v_r, b_k) configuration is semiregular of girth ≥ 6. A (v_r, b_k) configuration is *symmetric* if $v = b$ (which is equivalent to saying that $r = k$). A symmetric (v_k, v_k) configuration is called a (v_k) configuration.

3. Combinatorial and geometric configurations

Our definition of configuration is purely combinatorial, yet the origins of configurations come from geometry [10]. It is quite natural to represent configurations by geometric objects of two types, such as points and lines, points and curves, lines and planes, etc. From the abstract, combinatorial configurations we obtain concrete, geometric configurations. Geometric drawings are useful for depicting combinatorial configurations. They also carry information about Levi and Menger graphs. An n-lateral, a point-line-point-line $\cdots$ cycle in the configuration, corresponds to a $2n$ cycle in its Levi graph. Even Menger graph can be read off from the geometric picture. Configuration points correspond to the vertices while all edges are hidden in the lines (curves). Note that each pair of vertices on the same curve is adjacent in the Menger graph. A drawing of a (v_3) configuration provides a drawing of its Menger graph if we connect up the ends of each line to form a closed curve.

4. Duality

For each incidence structure $\mathcal{C} = (P, B, I)$ the *dual structure* is $\mathcal{C}^d = (P^d, B^d, I^d)$, where $P^d = B, B^d = P, I^d = I)$. Both $\mathcal{C}$ and $\mathcal{C}^d$ share the same Levi graph except that the black–white coloring of vertices is reversed. The Menger graph $G(\mathcal{C}^d)$ of $\mathcal{C}^d$ is known as the *dual Menger graph* $D(\mathcal{C})$ of $\mathcal{C}$. For example, Fig. 1 shows the *Pasch configuration* $(6_2, 4_3)$ which is also known as *complete quadrilateral* and its dual the *complete quadrangle* $(4_3, 6_2)$, their shared Levi graph and their Menger graphs.

5. Automorphisms and anti-automorphisms

If the incidence structure $\mathcal{C}$ is isomorphic to its dual $\mathcal{C}^d$, we say that it is self-dual and the isomorphism is called a *duality*. A duality of order 2 is called a *polarity*. An isomorphism of $\mathcal{C}$ to itself is called an automorphism or colinearity. Automorphisms of $\mathcal{C}$ form a group denoted by $\mathrm{Aut}_0\, \mathcal{C}$. We may consider automorphisms and dualities (anti-automorphisms) as acting on the disjoint union $P \cup B$. They together form the *extended group of automorphisms* $\mathrm{Aut}\, \mathcal{C}$. The Levi graph $L(\mathcal{C})$ of a configuration $\mathcal{C}$ is bipartite and carries complete information about the configuration. The extended automorphism group $\mathrm{Aut}\, \mathcal{C}$ coincides with the automorphism group of $L(\mathcal{C})$, while $\mathrm{Aut}_0\, \mathcal{C}$ is the subgroup which fixes the two bipartition sets setwise.

While the symmetries of configurations are perfectly mirrored in symmetries of the corresponding Levi graphs, the connection to symmetries of Menger graphs is less evident.

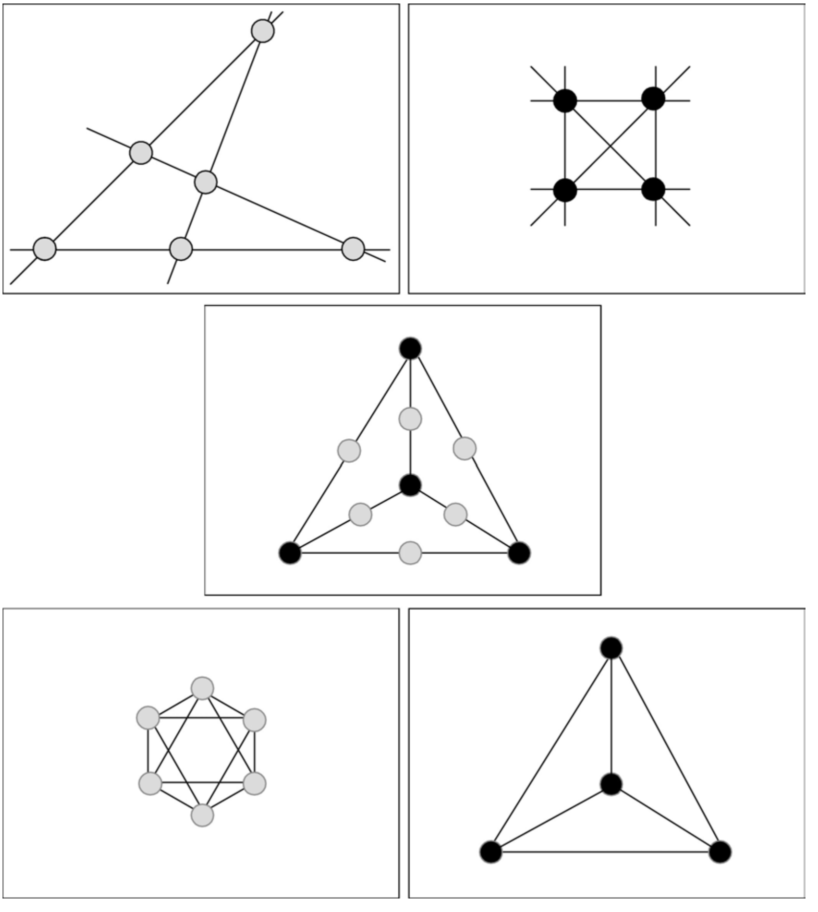

Fig. 1. The Pasch configuration, its dual, their Levi graph and their Menger graphs.

Clearly the group $\mathrm{Aut}_0\, \mathcal{C}$ acts on $M(\mathcal{C})$. It transpires that the action is faithful at least in the case of connected configurations.

6. The GRAY graph and the gray configuration

The smallest known cubic edge- but not vertex-transitive graph has 54 vertices and is known as the Gray graph, which we will call $\mathcal{G}$ in this paper [2, 3]. It is shown in Fig. 2. Since the Gray graph is bipartite, regular of valence 3, of girth 8, and edge but not vertex transitive, it is the Levi graph of a dual pair of symmetric (27_3) configurations [14], both of which are triangle-free and flag transitive but not self-dual. This is the smallest pair of such configurations. Drawings of these two configurations exhibiting rotational symmetry and their "polycyclic structure" are shown in Fig. 3; see [1, 9] for the theoretical background. These drawings illustrate a problem in straight-line realizations of configurations, that

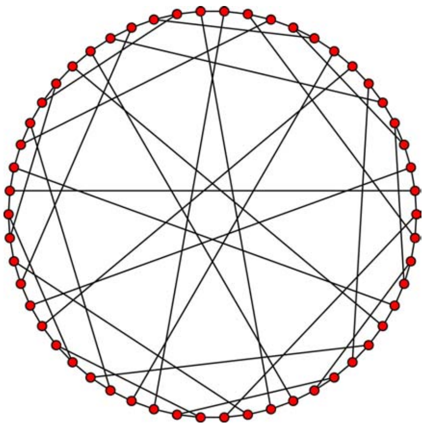

Fig. 2. The Gray graph as drawn by Milan Randić.

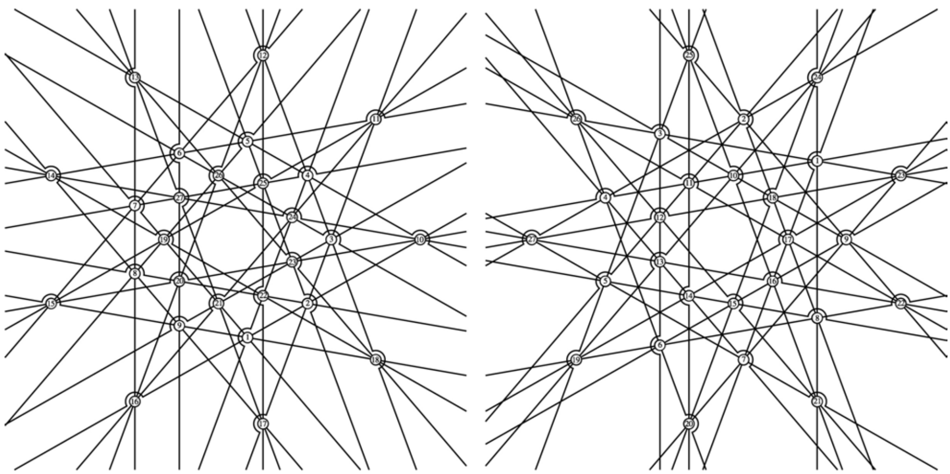

Fig. 3. Dual configurations arising from the Gray graph. Both drawings contain false incidences that are clearly marked. However, if we admit all incidences, we may view these figures as a pair of dual (27_4) configurations.

sometimes drawings contain false incidences. Following [4] we realize in Fig. 4 the first of these two as a cube-shaped configuration of 27 points in 27 lines in $\mathbb{R}^3$, and we will refer to this configuration as the Gray configuration. Using this drawing it becomes clear that the Menger graph M of the Gray configuration, shown in Fig. 5(a), is isomorphic to $K_3 \square K_3 \square K_3$. (Here we use the notation $\square$ of [12] to represent the Cartesian product of graphs.)

7. The genus embedding

Let $\gamma(G)$ denote the genus of the graph G. This parameter denotes the least integer k, such that G admits an embedding into an orientable surface of genus k. Several years ago

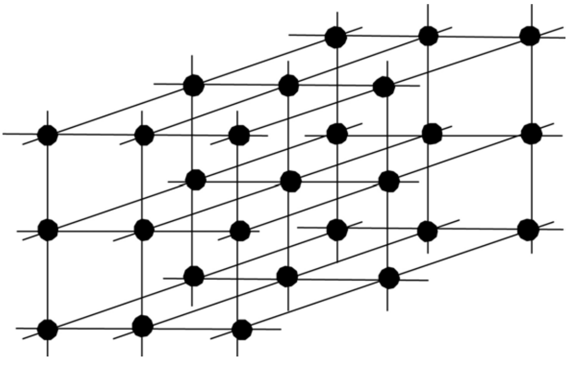

Fig. 4. Spatial version of the Gray configuration consists of 27 points and 27 lines grouped in three classes, each containing 9 parallel lines.

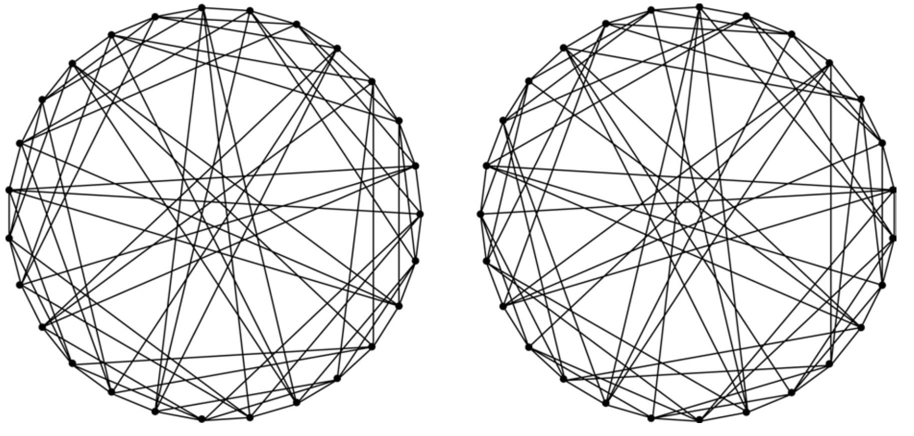

Fig. 5. Menger graph M on the left hand side and dual Menger graph D on the right hand side of the Gray configuration.

it was shown that $\gamma(K_3 \Box K_3 \Box K_3) = 7$. The genus embedding was constructed by Mohar et al. [15]. One nice feature of the genus embedding is that its dual is bipartite; i.e., its faces can be colored in two colors so that each edge separates faces of different colors. All of the faces of one color are triangles. These 27 triangles are the only triangles in the graph and correspond to lines in the configuration. The Gray graph thus admits an embedding into the surface of genus 7. If we keep the original vertices and introduce the centers of triangles as new vertices with an old vertex v adjacent to a new vertex t if and only if v lies on the boundary of the triangle t, the resulting graph is the Gray graph. Hence the Gray graph fits onto the same surface; see Fig. 6.

8. The lower bound

This shows that the upper bound for the genus is 7. The lower bound 7 follows from the following:

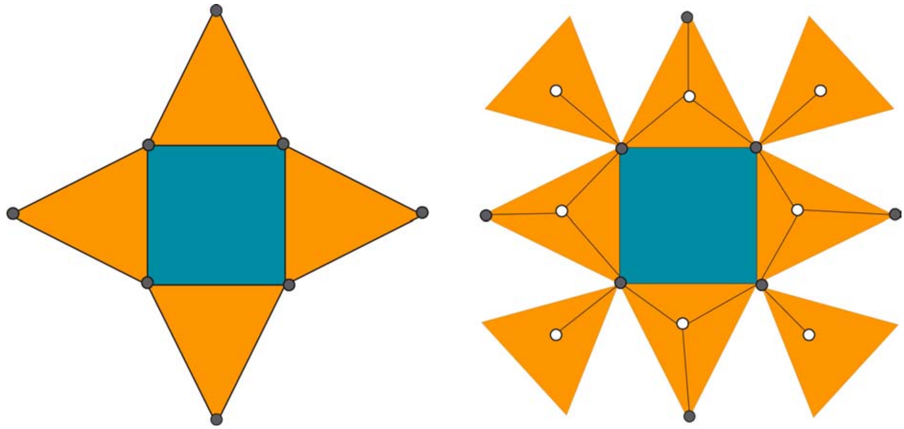

Fig. 6. The Gray graph fits into the same surface.

Proposition 1. *Let L be the Levi graph and let M be the Menger graph of some (v_3) configuration $\mathcal{C}$, then $\gamma(M) \le \gamma(L)$.*

Proof. Start with the genus embedding of the cubic bipartite graph L with vertices colored, say black and white. By the reverse process depicted in Fig. 6 one can obtain the embedding of M in the same surface. For each white vertex w of L having three adjacent black vertices, say a, b, c, introduce three new edges, forming a triangle that joins the three black vertices a, b, c. Remove all original edges and all white vertices.

9. The dual Menger graph D and the Holt graph

There is one more case to consider. Namely, the dual Menger graph D can also be embedded into the surface of genus 7. In order to understand its structure we interactively experimented with a powerful computer system Vega [16]. As a by-product of these experiments we were able to produce the drawings of Fig. 7. It turns out that this graph is quite interesting. It is the Cayley graph $\mathrm{Cay}(G, S)$, where Γ is the semidirect product

$$\Gamma = \mathbb{Z}_9 \rtimes \mathbb{Z}_3 = \langle a, b \mid a^9 = b^3 = 1, b^{-1}ab = a^4 \rangle$$

and $S = \{a, a^{-1}, ab, (ab)^{-1}, (ab^2), (ab^2)^{-1}\}$.

Therefore D can be described as a Z_9-covering graph over the base graph in Fig. 8. The three gray vertices can be identified with the three cosets of the cyclic subgroup generated by the element a. Letting $x = a, y = ab, z = ab^2$, it may be verified that the group Γ admits the presentation

$$D = \langle x, y, z \mid x^9 = y^9 = z^9 = 1, y^{-1}xy = x^4, x^{-1}yx = y^7, xyz = 1 \rangle.$$

Consequently we also have

$$\{x^{-1}yx = y^7, x^{-1}zx = z^4, z^{-1}xz = x^7, z^{-1}yz = y^4\}.$$

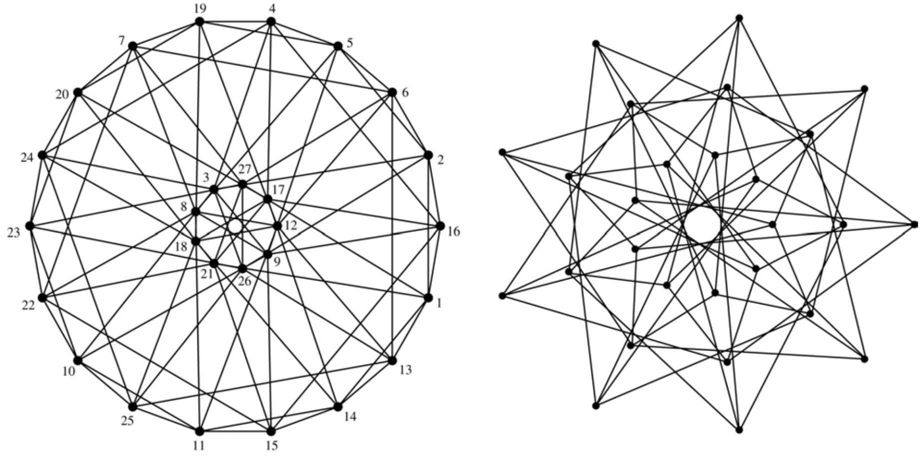

Fig. 7. The dual Menger graph D and the Holt graph H, its spanning subgraph.

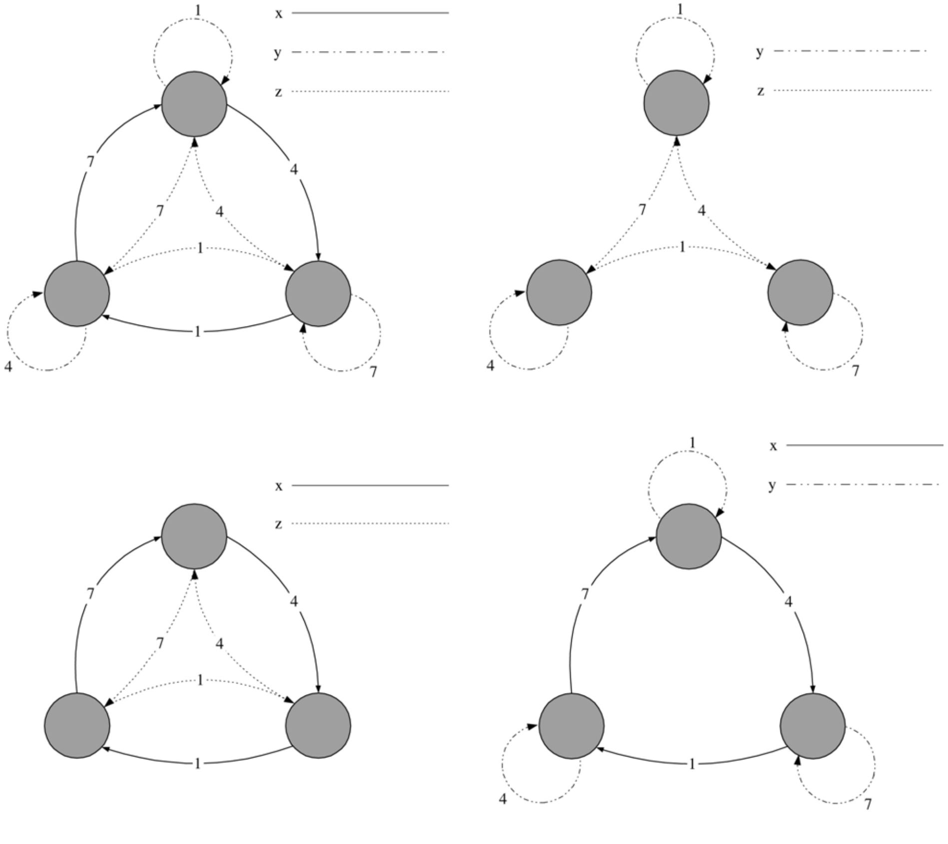

Fig. 8. The $\mathbb{Z}_9$ voltage graph for the dual Menger graph and the three voltage subgraphs for the Holt graph.

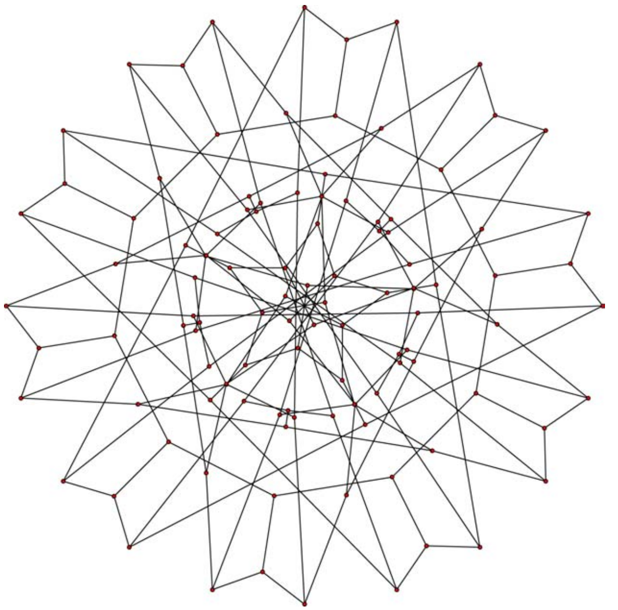

Fig. 9. The second semisymmetric cubic graph on 110 vertices is bipartite and has girth 10.

Note the cyclic symmetry of x, y, z in the presentation of Γ [11]. We remark that a deletion of any of the three 2-factors isomorphic to $3C_9$ corresponding to x, y, or z gives rise to a graph isomorphic to the Holt graph H. The 4-valent Holt graph of girth 5 is the smallest 1/2-arc transitive graph, that is, vertex- and edge- but not arc-transitive; see Fig. 7. Fig. 8 shows two essentially different ways to describe H as a $\mathbb{Z}_9$ cover graph of a graph on three vertices. In one case the base graph is a doubled triangle, in the other, it is a triangle with loops. In each case, the voltage assignment from $\mathbb{Z}_9$ is shown.

10. The final problems

What is the genus of D? What is the genus of H? The genus of $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ is known [6]. Namely, $\gamma(\mathbb{Z}_9 \rtimes \mathbb{Z}_3) = 4$. On the other hand we proved that D admits an embedding into the surface of genus 7, with not very high symmetry, as can be derived from the construction in [15]. Since H is a subgraph of D it follows that $4 \leq g(H) \leq g(D) \leq 7$. It would be an interesting research problem to find a (v_3) configuration for which the inequality in Proposition 1 is strict, or prove that the equality holds. In the latter case that would prove $g(D) = 7$. It might be worth pointing out that the minimal genus embedding [6] for this group comes from a different generating set than $\{x, y\}$, which is why [14] does not give an immediate answer to the genus of H. The Gray graph is the smallest semisymmetric cubic graph. There are others: the next largest one has 110 vertices [13] and is shown in Fig. 9. Since it is bipartite, of girth 10, the corresponding dual configurations may also be the clue to its genus. The genus question can also be asked for the third graph on 112 vertices [4, 7].

We conclude by explicitly stating a theorem whose proof is the essence of this paper.

Theorem 2. *The genus of the Gray graph is 7.*

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